TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 1

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Homework 1.1 (An example of propagation of convergence*)**.** Let $D \subset \mathbf{C}$ be a bounded domain and denote by $\bar{D}$ is closure. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of continuous functions $f_n\colon \bar{D} \to \mathbf{C}$ such that f_n is holomorphic in D for every $n \in \mathbf{N}$. Assume $(f_n)_{n\in\mathbf{N}}$ converges uniformly on the boundary ∂D.

a. Show $(f_n)_{n\in\mathbf{N}}$ converges uniformly on $\bar{D}$ to some function $f\colon \bar{D} \to \mathbf{C}$[1].
b. Show f is holomorphic on D.

Homework 1.2 (Sequences of holomorphic functions)**.**

a. Let $(f_n)_{n\in\mathbf{N}}$ constitute a sequence of holomorphic functions $f_n\colon B_1(0) \to \mathbf{C}$ converging locally uniformly to a given holomorphic function $f\colon B_1(0) \to \mathbf{C}$. Does the sequence $(f_n^{(n)})_{n\in\mathbf{N}}$ of derivatives of f_n with increasing order converge locally uniformly to a continuous function $g\colon B_1(0) \to \mathbf{C}$? Give a proof or find a counterexample.
b. Give an example of an open set $U \subset \mathbf{C}$ and a sequence $(f_n)_{n\in\mathbf{N}}$ of holomorphic functions $f_n\colon U \to \mathbf{C}$ that converges locally uniformly to a function $f\colon U \to \mathbf{C}$ and such that f_n has exactly one zero for every $n \in \mathbf{N}$ while f has no zero. Can one construct a counterexample under the requirement of uniform convergence on U?

Homework 1.3 (Convergence of varying path-integrals)**.** Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of holomorphic functions $f_n\colon U \to \mathbf{C}$ that converges locally uniformly on a given open set $U \subset \mathbf{C}$ to some function $f\colon U \to \mathbf{C}$. Moreover, let $(\gamma_n)_{n\in\mathbf{N}}$ be a sequence of C^1-paths $\gamma_n\colon [0,1] \to U$ such that $\gamma_n \to \gamma$ and $\gamma_n' \to \gamma'$ uniformly on $[0,1]$ as $n \to \infty$, where $\gamma\colon [0,1] \to U$ is a C^1-path. Show

$$\lim_{n\to\infty} \int_{\gamma_n} f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Homework 1.4 (Osgood's theorem)**.** Let $U \subset \mathbf{C}$ be open and $(f_n)_{n\in\mathbf{N}}$ be a sequence of holomorphic functions $f_n\colon U \to \mathbf{C}$ that converges pointwise to $f\colon U \to \mathbf{C}$. The goal of this exercise is to show there exists an open, dense subset $U_0 \subset U$ such that $(f_n)_{n\in\mathbf{N}}$ is locally uniformly bounded on U_0. As we will see in the course, this implies the local uniform convergence of $(f_n)_{n\in\mathbf{N}}$ to f on U_0; in particular, f is holomorphic on U_0.

a. Define the set of points where $(f_n)_{n\in\mathbf{N}}$ is locally uniformly bounded, i.e.,
$$U_0 := \{z \in U : \text{there exists } r > 0 \text{ with } \sup_{n\in\mathbf{N}} \sup_{z'\in B_r(z)} |f_n(z')| < \infty\}.$$
Show U_0 is open.
b. Show if U_0 is not dense in U, then there exists a ball $B_{r_0}(z_0) \subset U$ such that for all balls $B_{r'}(z') \subset B_{r_0}(z_0)$ the sequence $(f_n)_{n\in\mathbf{N}}$ is not uniformly bounded on $B_{r'}(z')$.
c. Use b. to construct a sequence of nested closed balls $\bar{B}_{r_k}(z_k) \subset B_{r_{k-1}}(z_{k-1})$ and a subsequence $(f_{n_k})_{k\in\mathbf{N}}$ such that $|f_{n_k}| \geq k$ on $\bar{B}_{r_k}(z_k)$.
d. Show the intersection of the sequence of closed balls from c. is nonempty. Derive a contradiction to the pointwise convergence of $(f_n)_{n\in\mathbf{N}}$ and conclude the proof.

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[1]**Hint.** Show $(f_n)_{n\in\mathbf{N}}$ is a Cauchy sequence with respect to uniform convergence.